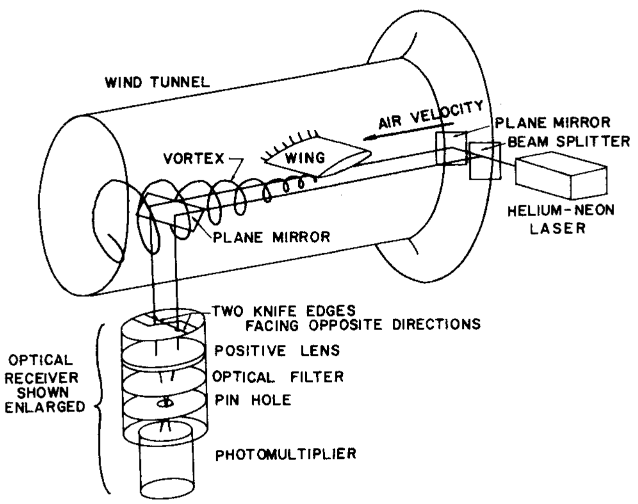

Fig. 1 Schematic arrangement of the dual beam laser vortex detection system in the low speed wind tunnel.

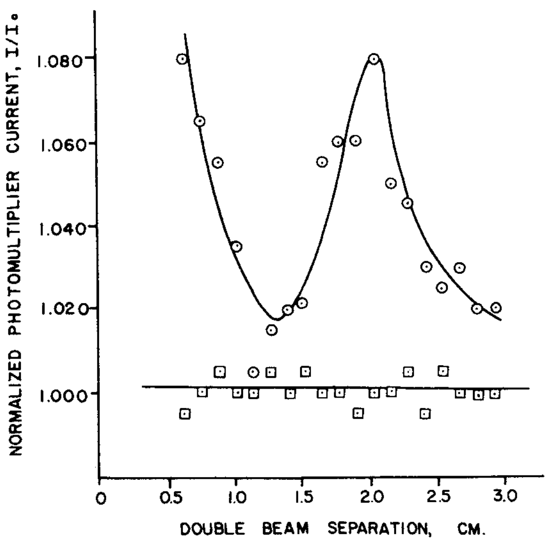

Fig. 2 Comparison of the vortex detection system response to a wing generated vortex with a thermal plume in the absence of a vortex. ⊙ = with vortex from wing at 10° angle of attack, $V = 13.7$ m/sec. □ = without wing but with hot plate under dual beams, $V = 0$.

light reaching the photomultiplier. If one beam is unaffected but the other is deflected in the vertical plane, a smaller net change in the photomultiplier current would result.

In full-scale operation over a runway, the laser beams should propagate above and parallel to the runway. Existing runway approach light towers on both ends of the runway could be used to house the necessary system of mirrors. This would not, therefore, add any new protruding structures adjacent to the runway. The beam separation and height above ground are variables which must be determined for best system response to a vortex. For safety, the laser wavelength should be selected such that the light beams would not transmit through the aircraft windshields. The laboratory tests described above indicate that, in principle, a full-scale system of this type should be able to detect wingtip vortices over a runway.

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Approximate Solution for Minimum Induced Drag of Wings with Given Structural Weight

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Introduction

In recent years great achievements have been obtained in reducing different kinds of drag for aircraft flying at high subsonic or transonic speeds. Surprisingly little attention has been paid, however, to the problem of minimizing induced drag. Elliptic spanwise loading has been known for a long time to be optimum in this respect if overall lift and wing span are given. In practical applications, the span cannot be chosen at will, but is restricted from structural considerations. Hence the spanwise loading that provides minimum induced drag in steady flight will be determined by overall lift, and by the weight of the wing. This weight changes during the flight as fuel is extracted from the tanks, and it is therefore very difficult to specify an auxiliary condition that holds true for the mathematical formulation of the optimization problem. If we take wing structural weight alone as the decisive factor, the problem can be solved, provided a relation is found between this weight and the relevant aerodynamic parameters. Such a solution was given by L. Prandtl[1]. He based it on the assumption that a direct proportionality exists between the weight of the spars and the local bending moment. Recently A. Klein and S. P. Viswanathan[2] worked out a solution for the case where the wing-root bending moment is prescribed. In this Note a solution is derived that is believed to represent an even better approximation for the optimization problem. It is based on the common practice to determine the structural weight of such wings for steady flight by the integrals of the spanwise shear-force and bending-moment distributions.

Formulation of the Problem

The integrals of the spanwise distributions of shear-force F and bending-moment M that are due to the airload are

$$\int_0^s F(y)dy = \frac{1}{2}\rho_\infty V_\infty^2 \int_0^s \int_y^s c(y')C_L(y')dy'dy$$

and

$$\int_0^s M(y)dy = \frac{1}{2}\rho_\infty V_\infty^2 \int_0^s \int_y^s c(y')C_L(y')(y'-y)dy'dy$$

Received February 25, 1974; revision received September 26, 1974.
Index category: Aircraft Aerodynamics (Including Component Aerodynamics).

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where y is the spanwise coordinate, y' its integration variable, s wing semi-span, c wing chord, ρ_∞ and V_∞ the density and speed of the undisturbed flow, and C_L the sectional lift coefficient. It is easily shown that

$$\int_0^s F(y)dy = \frac{1}{2}\rho_\infty V_\infty^2 \int_0^s c(y)C_L(y)y\,dy,$$

$$\int_0^s M(y)dy = \int_0^s F(y)y\,dy =$$

$$\frac{1}{4}\rho_\infty V_\infty^2 \int_0^s c(y)C_L(y)y^2 dy$$

We assume that for steady-flight conditions the wing structural weight is approximately determined if these two integrals are given together with overall lift. We want to determine the spanwise loading which makes the induced drag of a wing with prescribed structural weight a mimimum. The necessary auxiliary conditions can be stated now from the foregoing equations. For convenience we will introduce the non-dimensional coordinate $\eta = y/s$ and the dimensionless circulation $\gamma(\eta) = c(\eta)C_L(\eta)/4s$. Then we have to postulate that

$$L = 2\rho_\infty V_\infty^2 s^2 \int_0^1 \gamma(\eta)d\eta \tag{1}$$

$$s\int_0^1 F(\eta)d\eta = 2\rho_\infty V_\infty^2 s^3 \int_0^1 \gamma(\eta)\eta d\eta \tag{2}$$

$$s\int_0^1 M(\eta)d\eta = \rho_\infty V_\infty^2 s^4 \int_0^1 \gamma(\eta)\eta^2 d\eta \tag{3}$$

are given quantities. The problem will be solved in two steps. First of all one has to determine the spanwise loadings $\gamma(\eta)$-for which the induced drag D_i,

$$D_i = 2\rho_\infty V_\infty^2 s^2 \int_0^1 \gamma(\eta)\alpha_i(\eta)d\eta \tag{4}$$

where $\alpha_i(\eta)$ is the spanwise distribution of the flow angle induced by $\gamma(\eta)$ far behind the wing, and attains a minimum for any span chosen. Thereafter that particular optimum solution $\gamma_{opt}(\eta)$ has to be selected which provides the absolute minimum $(D_i)_{\min}$.

The first part of the problem can be tackled by application of variational principles. Applying the well-known mutual drag theorem, solve for

$$\delta D_i = 2\int_0^1 \delta\gamma(\eta)\alpha_i(\eta)d\eta = 0 \tag{5}$$

with the auxiliary conditions

$$\int_0^1 \delta\gamma(\eta)d\eta = 0 \tag{6}$$

$$\int_0^1 \delta\gamma(\eta)\eta d\eta = 0 \tag{7}$$

$$\int_0^1 \delta\gamma(\eta)\eta^2 d\eta = 0 \tag{8}$$

Comparison of Eqs. (5) to (8) reveals that

$$\alpha_i(\eta) = C_1 + C_2|\eta| + C_3\eta^2 \tag{9}$$

Hence the required spanwise downwash-distribution is parabolic. The constants C_1, C_2, and C_3 are arbitrary, but they are determined so that $\gamma(\eta)$ will satisfy the constraints of Eqs. (1) to (3). Then a unique solution to the variational problem is obtained. Since the spanwise loading and the downwash are related by Biot-Savart's law, the present problem reduces to solving the integral equation

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{d\gamma(\eta')}{d\eta'}\frac{d\eta'}{\eta-\eta'} = C_1 + C_2|\eta| + C_3\eta^2 \tag{10}$$

Solution and Discussion of Result

Equation (10) has a closed solution which is established in a way similar to that described in Ref. 3. It reads

$$\frac{d\gamma}{d\eta} = \frac{-2}{\pi}\left(\frac{1-\eta}{1+\eta}\right)^{1/2}\int_{-1}^{1}(C_1 + C_2|\eta'| + C_3\eta'^2)\left(\frac{1+\eta'}{1-\eta'}\right)^{1/2}\frac{d\eta'}{\eta-\eta'} + \frac{2k}{(1-\eta^2)^{1/2}} \tag{11}$$

where k is an arbitrary constant determined from the fact that $\gamma(\eta) = \gamma(-\eta)$. Using relations in Refs. 3 and 4,

$$\gamma(\eta) = (1-\eta^2)^{1/2}\left(2C_1 + \frac{2C_2}{\pi} + C_3\right) - \frac{C_2}{\pi}\eta^2 \ln 1-(1-\eta^2)/1+(1-\eta^2)^{1/2} - \frac{2}{3}C_3((1-\eta^2)^3)^{1/2} \tag{12}$$

The constants are determined by the values given by Eqs. (1)—(3). Equation (12) defines the family of spanwise loadings that ensures minimum induced drag for any chosen span.

To make our solution more comprehensive, we will compare it to that of a wing with elliptic loading of span s_e which produces the same lift at the same dynamic pressure $\rho_\infty V_\infty^2/2$ and has equal structural weight. The constants C_1, C_2, and C_3 can then be computed by introducing $\gamma(\eta)$ from Eq. (12) into Eqs. (1)—(3) and equating these relations to those for elliptic loading. The values of the various integrals are either available in tables or can be determined by integrating by parts. With the help of the integrals the three constants are obtained in terms of the ratio $\sigma = s/s_e$ between the spans of the two wings. The result is

$$C_1 = \frac{\gamma Re}{2\sigma^4}(36\sigma^2 - 80\sigma + 45) \tag{13}$$

$$C_2 = -\frac{15\pi}{\sigma^4}\gamma_{Re}(\sigma-1)(2\sigma-3) \tag{14}$$

$$C_3 = \frac{30}{\sigma^4}\gamma_{Re}(\sigma-1)(3\sigma-5) \tag{15}$$

The circulation γ_{Re} at the root of the wing with elliptic loading is related to the root circulation γ_R of the wing under consideration by

$$\frac{\gamma_R}{\gamma_{Re}} = \frac{1}{\sigma^4}(6\sigma^2 - 10\sigma + 5) \tag{16}$$

On substitution of the relations Eqs. (13)—(16) into Eqs. (12), (9), and (4), there is obtained

$$\gamma(\eta) = \frac{\gamma_R}{6\sigma^2 - 10\sigma + 5}[(66\sigma^2 - 170\sigma + 105)(1-\eta^2)^{1/2} + 15(\sigma-1)(2\sigma-3) \times \eta^2 \ln\{[1-(1-\eta^2)^{1/2}]/[1+(1-\eta^2)^{1/2}]\} - 20(\sigma-1) \times (3\sigma-5)(1-\eta^2)^{3/2}] \tag{17}$$

$$\alpha_i(\eta)=\frac{\gamma_R}{6\sigma^2-10\sigma+5}[30(\sigma-1)(3\sigma-5)\eta^2-15\pi(\sigma-1)\times(2\sigma-3)|\eta|+18\sigma^2-40\sigma+\frac{45}{2}] \quad (18)$$

$$\frac{D_i}{D_{ie}}=\frac{4}{\sigma^6}(9\sigma^4-40\sigma^3+\frac{145}{2}\sigma^2-60\sigma+\frac{75}{4}) \quad (19)$$

where D_{ie} is the induced drag of the wing with elliptic loading. The second part of the problem, determining the absolute minimum for D_i/D_{ie}, is easily solved now. It is found to be

$$\left(\frac{D_i}{D_{ie}}\right)_{min}=0.929 \text{ with } \sigma_{opt}=1.160$$

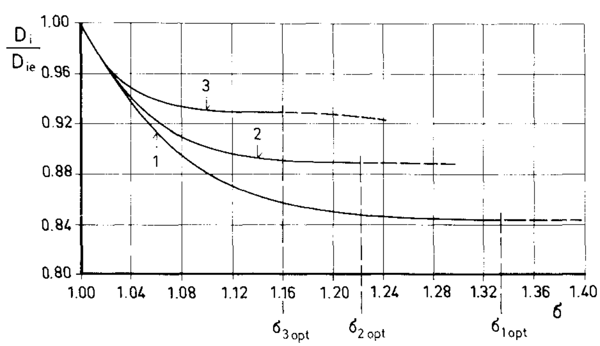

Fig. 1 Induced drag vs span-ratio: curve 1—solution according to Ref. 2; curve 2—solution according to Ref. 1; curve 3—present solution.

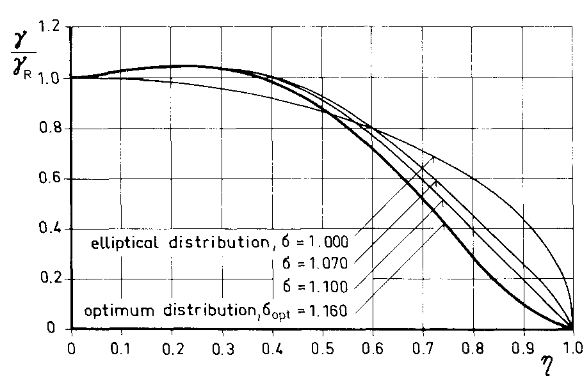

Fig. 2 Optimum spanwise loading distribution.

The respective optimum spanwise loading is

$$\gamma_{opt}(\eta)=\gamma_R[3.3008((1-\eta^2)^3)^{1/2}-2.3008\times(1-\eta^2)^{1/2}-1.1075\,\eta^2\log_e\frac{1-(1-\eta^2)^{1/2}}{1+(1-\eta^2)^{1/2}}] \quad (20)$$

Hence a wing with a circulation-distribution according to Eq. (20) will have about 7% less induced drag than an elliptically loaded wing with the same structural weight and lift, if its span is 16% larger.

For a prescribed lift, different auxiliary conditions give different optimum spanwise downwashes and loadings that provide minimum induced drag, viz; a) If the span is specified, C_2 and C_3 vanish, the downwash is constant and the loading is elliptic. b) If the integral of the spanwise shear-force distribution is specified, C_3 vanishes, the downwash varies linearly along the span and the loading is as given in Ref. 2. c) If the integral of the spanwise bending-moment distribution is specified, C_2 vanishes, the downwash is parabolic, $\alpha_i(\eta)=f_1(\eta^2)$, and the loading is as given in Ref. 1. d) If both the integrals of the spanwise bending-moment and shear-force distributions are prescribed, the downwash is parabolic, $\alpha_i(\eta)=f_2(\eta,\eta^2)$, and the loading is that according to Eq. (20). Note that one can show that the wing-root bending moment used in Ref. 2 and the moment of inertia used in Ref. 1 are identical with the integrals of the spanwise shear-force and bending-moment distributions, respectively.

Figure 1 compares the relationship between the induced drag D_i/D_{ie} and the span-ratio σ which was derived in Refs. 1 and 2 with that of Eq. (19) in the present Note. The minima of the curves are actually points of inflexion, but the solutions have no physical meaning for $\sigma>\sigma_{opt}$ as was explained by Prandtl.[1] In Fig. 2 the optimum spanwise loading for $\sigma_{opt}=1.16$ according to Eq. (20) has been plotted along with elliptic distribution and two more selected loadings of the family specified by Eq. (17) for $\sigma=1.10$ and 1.07. Induced drag in the last case is only 0.8% larger than the minimum value. Drag rises fast, however, if distributions $\gamma(\eta)$ for still smaller span-ratios are chosen.

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